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Generalized Pseudospectral Methods for Modeling and Reverse-time Migration in Orthorhombic Media

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SUMMARY

We derive separable approximations to orthorhombic dispersion relations that allow accurate and efficient modeling of P-waves. These allow extension to orthorhombic media of generalized pseudospectral methods previously used for modeling and reverse-time migration of P-waves in transversely isotropic media.
Introduction

Current seismic imaging techniques routinely handle wave propagation in transversely isotropic (TI) media. Oriented stresses or fracturing can cause additional azimuthal variations in seismic properties, lowering the material symmetry and requiring the use of orthorhombic anisotropy. Azimuthal velocity and amplitude variations can cause complications for achieving optimal seismic imaging, but can also present opportunities to extract important reservoir-related fracture properties.

Etgen and Brandsberg-Dahl (2009) introduced a generalization of pseudospectral methods for modeling and reverse-time migration in TI media. Their method depended on deriving accurate separable approximations to dispersion relations. We show here how to derive useful separable approximations to orthorhombic P-wave dispersion relations, and then how to use them to extend generalized pseudospectral methods.

Separable dispersion relation approximations and generalized pseudospectral extrapolation

Suppose one has a dispersion relation of the form \( \omega^2 = A(k, v) \), where \( \omega \) is frequency, \( k \) is the wavenumber vector, and \( v \) represents the material or velocity parameters. Transforming from the frequency domain back to the time domain gives

\[
\frac{\partial^2 u}{\partial t^2} = A(k, v) u . \tag{1}
\]

Suppose further that the dispersion relation can be written in the separable form

\[
A(\mathbf{v}(\mathbf{x})) = \sum x f(\mathbf{v}(\mathbf{x})) g(\mathbf{k}) .
\]

Then, after replacing the time derivative by a finite difference approximation, equation (1) can be implemented for time extrapolation as

\[
u(\mathbf{x}, t + \Delta t) \approx 2u(\mathbf{x}, t) - u(\mathbf{x}, t - \Delta t) - \Delta t^2 \sum f(\mathbf{v}(\mathbf{x})) FT^{-1}[g(\mathbf{k})FT[u(\mathbf{x}, t)]] , \tag{2}
\]

where \( \mathbf{x} = (x, y, z) \) is spatial location, \( t \) is time, and \( FT \) indicates a 3D spatial Fourier transform. This is now in the form of a pseudospectral algorithm, but generalized in the sense that now some of the wavenumber functions \( g \) may not correspond to conventional space derivatives. This method was introduced for TI media by Etgen and Brandsberg-Dahl (2009).

From the (axis-aligned) orthorhombic Christoffel equations one gets a dispersion equation

\[
0 = \omega^6 - a\omega^4 + b\omega^2 - c , \tag{3}
\]

with the coefficients

\[
a = -\left(H_{11} + H_{22} + H_{33}\right) ,
\]

\[
b = H_{11}H_{22} + H_{11}H_{33} + H_{22}H_{33} - H_{12}^2 - H_{13}^2 - H_{23}^2 ,
\]

\[
c = H_{11}H_{23}^2 + H_{22}H_{31}^2 + H_{33}H_{12}^2 - H_{11}H_{32}H_{33} - 2H_{12}H_{23}H_{13} .
\]

The orthorhombic Christoffel matrix elements \( H_{ij} \) here are given in terms of the density normalized stiffnesses \( a_{ij} \equiv c_{ij} / \rho \) and spatial wavenumbers \( k_x, k_y, \) and \( k_z \) as

\[
H_{11} = a_{11}k_x^2 + a_{22}k_y^2 + a_{33}k_z^2 ;
\]

\[
H_{12} = (a_{12} + a_{26})k_xk_y ;
\]

\[
H_{13} = (a_{13} + a_{35})k_xk_z ;
\]

\[
H_{23} = (a_{33}k_x^2 + a_{22}k_y^2 + a_{44}k_z^2) ;
\]

\[
H_{22} = (a_{44}k_y^2 + a_{22}k_x^2 + a_{33}k_z^2) ;
\]

\[
H_{33} = (a_{33}k_z^2 + a_{33}k_y^2 + a_{44}k_z^2) .
\]

A solution to equation (3) corresponding to P waves is given (Tsvankin, 1997) by

\[
\omega^2 = 2\mu \left[ \frac{1}{3} \left( b - \frac{2}{3} \right) - \frac{a}{3} \right] , \tag{6}
\]

where
\[ \mu = \cos \left( \frac{1}{3} \cos^{-1} \left[ \frac{\sqrt{27}}{2} \left( \frac{a}{3} \right)^3 - \frac{ab}{3} + c \left( b - \frac{a^2}{3} \right)^{3/2} \right] \right) . \] (7)

The orthorhombic P-wave dispersion relation in equation (6) can be approximated as
\[ \omega^2 = -\frac{a}{2} \left\{ 1 + \sqrt{1 - 4\gamma} \right\} = -a \left\{ 1 - \gamma - \gamma^2 - 2\gamma^3 - 5\gamma^4 - 14\gamma^5 - 42\gamma^6 - \cdots \right\} , \] (8)
with \( \gamma \equiv b / a^2 \). With little loss of accuracy, the convergence of this series can usually be greatly improved by using pseudo-acoustic approximations. We can change variables to:

\[
\begin{align*}
v_{px}^2 &= a_{11} ; & v_{py}^2 &= a_{44} ; & v_{pz}^2 &= \left[ a_{23} (a_{23} + a_{44}) + a_{44} (a_{23} + a_{33}) \right] / (a_{33} - a_{44}) ; \\
v_{py}^2 &= a_{22} ; & v_{pz}^2 &= a_{55} ; & v_{pm}^2 &= [a_{13} (a_{13} + a_{55}) + a_{55} (a_{13} + a_{33})] / (a_{33} - a_{55}) ; \\
v_{pc}^2 &= a_{33} ; & v_{ps}^2 &= a_{66} ; & v_{pm}^3 &= \left[ a_{12} (a_{12} + a_{66}) + a_{66} (a_{12} + a_{11}) \right] / (a_{11} - a_{66}) ,
\end{align*}
\] (9)

and then let \( v_{s1} \to 0 \), \( v_{s2} \to 0 \), and \( v_{s3} \to 0 \), giving the pseudo-acoustic Christoffel elements

\[
\begin{align*}
H_{11} &= v_{px}^2 k_x^2 ; & H_{12} &= v_{px} v_{pm} k_x k_y ; \\
H_{22} &= v_{py}^2 k_y^2 ; & H_{13} &= v_{pc} v_{pm} k_x k_z ; \\
H_{33} &= v_{pc}^2 k_z^2 ; & H_{23} &= v_{pc} v_{pm} k_y k_z .
\end{align*}
\] (10)

This then gives coefficients of the simpler form
\[
\begin{align*}
a &= -\left[ v_{px}^2 k_x^2 + v_{py}^2 k_y^2 + v_{pc}^2 k_z^2 \right] ; \\
b &= v_{px}^2 \left( v_{py}^2 - v_{pm}^2 \right) k_x^2 k_y^2 + v_{pc}^2 \left( v_{py}^2 - v_{pm}^2 \right) k_z^2 k_y^2 + v_{pc}^2 \left( v_{px}^2 - v_{pm}^2 \right) k_z^2 k_x^2 .
\end{align*}
\] (11)

The expansion in equation (8) is still not fully separable because of the powers of \( a \) in the denominators of each factor of \( \gamma \). We can choose a suitable reference velocity \( v_r \) and expand
\[
\begin{align*}
a^{-1} &= -v_r^2 k_r^2 \left[ 1 + 2k_r^2 E \right]^{-1} = -v_r^2 k_r^2 \left\{ 1 - 2k_r^2 E + 4k_r^4 E^2 - 8k_r^6 E^3 + 16k_r^8 E^4 - \cdots \right\} ,
\end{align*}
\] (13)

where \( k_r \equiv k_x^2 + k_y^2 + k_z^2 \) and \( E = -\frac{1}{2} \left( \frac{a}{v_r^2} + k_r^2 \right) \). Substituting equation (13) into equation (8) then gives a fully separable series expansion. The first terms yield the simple approximation:
\[ \omega^2 \approx v_{px}^2 k_x^2 + v_{py}^2 k_y^2 + v_{pc}^2 k_z^2 - \frac{v_{px}^2 \left( v_{py}^2 - v_{pm}^2 \right) k_x^2 k_y^2}{v_r^2} k_r^2 + \frac{v_{pc}^2 \left( v_{py}^2 - v_{pm}^2 \right) k_z^2 k_y^2}{v_r^2} k_r^2 . \] (14)

Higher order terms become substantially more complicated. Note that the first series in equation (8) expands the anellipticity \( \gamma \) around the ellipsoidal term \( a \). The second series in equation (13) then expands the ellipsoid \( a \) as a perturbation \( E \) around a reference sphere with radius \( v_r k_r \), suggesting that \( v_r \) should be chosen as some average P-wave velocity to get best convergence. We have found that \( v_r = \sqrt{v_{px} v_{py} v_{pc}} \) usually works well in practice.

For mathematical convenience so far we have assumed that the orthorhombic symmetry planes were aligned with the coordinate axes. For general use one needs to handle rotations with arbitrary orientations. Given a coordinate rotation \( \mathbf{\hat{x}} = \mathbf{b} \mathbf{x} \) defined by a \( 3 \times 3 \) orthogonal matrix \( \mathbf{b} \), one can rotate the corresponding spatial wavenumbers using the adjoint matrix \( \mathbf{\hat{k}} = \mathbf{b}^T \mathbf{k} \) and then use the separable approximations above, now in terms of the rotated wavenumbers.
Examples

Figure 1 shows a time snapshot of point-source modeling in a homogeneous axis-aligned orthorhombic medium using the generalized pseudospectral method of equation (2) based on a separable pseudo-acoustic dispersion approximation. The medium is homogeneous, with parameters (in km/s) \( v_{px} = 3.674 \), \( v_{py} = 4.135 \), \( v_{pc} = 3.0 \), \( v_{pm1} = 3.550 \), \( v_{pm2} = 2.683 \), \( v_{pm3} = 3.674 \), \( v_{x1} = 1.368 \), \( v_{x2} = 1.499 \), and \( v_{x3} = 1.2 \). One can see an accurate P-wave arrival, and complete elimination of shear wave energy. Figure 2 shows the same modeling result, but now in a rotated and tilted orthorhombic medium. Figure 3 shows similar generalized pseudospectral modeling in a complicated model with a complex salt body embedded in tilted and rotated orthorhombic sediments.

Discussion and Conclusions

The generalized pseudospectral method described here provides a simple, stable method for modeling and reverse-time migration of P-waves in mild orthorhombic anisotropy. Retaining good accuracy for stronger anisotropy requires additional, more complicated terms that increase the computational expense, and the approximations used here can fail when the anisotropy becomes sufficiently large. Implementation of these mixed-domain algorithms is most practical on computer architectures that efficiently implement large 3D FFTs.

Note that VTI is a special case of orthorhombic, so the pseudo-acoustic approximations here can be reduced to corresponding ones for TI by setting \( \frac{v_{pm3}}{v_{pm2}} = \frac{v_{py}}{v_{px}} \) and \( \frac{v_{pm3}}{v_{pm1}} = \frac{v_{pc}}{v_{pm1}} \). Equation (8) then reduces to the VTI approximation used by Alkhalifah (1998). If one additionally chooses \( v_{r} = v_{pc} \), equation (14) reduces to Harlan’s (1995) approximation as used by Etgen and Brandsberg-Dahl (2009). Note also that equation (2) really implements the leading term in the more accurate operator expansion (Etgen and Brandsberg-Dahl, 2009)

\[
\frac{u(x,t+\Delta t)}{\Delta t} = 2 \cos(\sqrt{\Omega} \Delta t) u(x,t) - u(x,t-\Delta t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \Omega^{2j} / \Delta t^{2j} u(x,t) - u(x,t-\Delta t) .
\]

This operator series can be efficiently approximated by Chebyshev polynomials (Pestana and Stoffa, 2010), or by using numerically fitted coefficients (Soubaras and Zhang, 2008; Fowler et al., 2010). These higher-order series methods allow using larger time steps, but still depend on separable approximations such as those introduced here to evaluate the operator \( \Omega^2 = A(k_v) \).

References

Figure 1 Time snapshot of a wavefield modeled with a generalized pseudospectral method in an axis-aligned orthorhombic medium. The pressure source is in the middle of the grid. We show a horizontal x-y slice through the source location (left), a vertical x-z slice (middle) and a vertical y-z slice (right).

Figure 2 Time snapshot of a wavefield modeled with a generalized pseudospectral method in a rotated and tilted orthorhombic medium. The pressure source is in the middle of the grid. We show a horizontal x-y slice through the source location (left), a vertical x-z slice (middle) and a vertical y-z slice (right).

Figure 3 Time snapshot of a wavefield modeled with a generalized pseudospectral method in a 3D model with a complex salt body embedded in highly heterogeneous tilted and rotated orthorhombic sediments.