Generalized pseudospectral methods for orthorhombic modeling and reverse-time migration
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Summary

We derive separable approximations to anisotropic dispersion relations that allow accurate and efficient modeling of P-waves. These enable extension to orthorhombic media of generalized pseudospectral methods previously used for modeling and reverse-time migration of P-waves in transversely isotropic media.

Introduction

Current seismic imaging techniques routinely handle wave propagation in transversely isotropic (TI) media. Oriented stresses or fracturing can cause additional azimuthal variations in seismic properties, lowering the material symmetry and requiring the use of orthorhombic anisotropy.

Etgen and Brandsberg-Dahl (2009) introduced a generalization of pseudospectral methods for modeling and reverse-time migration in TI media. Their method depended on deriving accurate separable approximations to dispersion relations. We show here how to derive useful separable approximations to orthorhombic P-wave dispersion relations, and then how to use them to extend generalized pseudospectral methods.

Separable dispersion relation approximations and generalized pseudospectral extrapolation

Suppose one has a dispersion relation of the form

\[ \omega = A(\mathbf{k}, \mathbf{v}) \]

where \( \omega \) is frequency, \( \mathbf{k} \) is the wavenumber vector, and \( \mathbf{v} \) represents the material or velocity parameters. Transforming from the frequency domain back to the time domain gives

\[ \frac{\partial^2 u}{\partial t^2} = -A(\mathbf{k}, \mathbf{v}) u . \]  

Suppose further that the dispersion relation can be written in the separable form \( A = \sum f_j(\mathbf{v}(x))g_j(\mathbf{k}) \). Then, after replacing the time derivative by a finite difference approximation, equation (1) can be implemented for time extrapolation as

\[ u(x,t+\Delta t) = 2u(x,t) - u(x,t-\Delta t) - \Delta t^2 \sum_j f_j(\mathbf{v}(x))FT^{-1}[g_j(\mathbf{k})FT[u(x,t)]] \]

where \( x = (x, y, z) \) is spatial location, \( t \) is time, and \( FT \) indicates a 3D spatial Fourier transform. This is now in the form of a pseudospectral algorithm, but generalized in the sense that now some of the wavenumber functions \( g \) may not correspond to conventional space derivatives. This method was introduced for TI media by Etgen and Brandsberg-Dahl (2009).

From the (axis-aligned) orthorhombic Christoffel equations one gets a dispersion equation of the form

\[ 0 = \omega^3 + a\omega^2 + b\omega + c , \]

with the coefficients

\[ a = -(H_{11} + H_{22} + H_{33}) \]
\[ b = H_{11}H_{22} + H_{11}H_{33} + H_{22}H_{33} - 2H_{12}H_{33} \]
\[ c = H_{11}H_{22}H_{33} - 2H_{12}H_{23}H_{33} . \]

The Christoffel matrix elements \( H_v \) here are given in terms of the density normalized stiffnesses \( a_i \equiv c_i / \rho \) and spatial wavenumbers \( k_x, k_y, \) and \( k_z \) as

\[ H_{11} = a_{11}k_x^2 + a_{12}k_y^2 + a_{13}k_z^2 \]
\[ H_{12} = a_{12}k_x^2 + a_{22}k_y^2 + a_{23}k_z^2 \]
\[ H_{13} = a_{13}k_x^2 + a_{33}k_z^2 \]
\[ H_{22} = (a_{12} + a_{22})k_y^2 \]
\[ H_{23} = (a_{13} + a_{33})k_z^2 \]

A solution to equation (3) corresponding to P waves is given (Tsvankin, 1997) by

\[ \omega^3 = 2\mu \left[ \frac{1}{3} \left( \frac{a^2}{3} - b \right) - \frac{a}{3} \right] , \]

where

\[ \mu = \cos \left( \frac{1}{3} \cos^{-1} \nu \right) \]
\[ \nu = -\sqrt{\frac{27}{2}} \left( \frac{a}{3} - \frac{b}{3} + c \right) \left( \frac{a^2}{3} - b \right)^{3/2} . \]

The P-wave dispersion relation in equation (6) can be expanded as a series in inverse powers of the coefficient \( a \):
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\[ \omega' = -a \left\{ 1 - \alpha + \beta - \alpha' + 3\alpha\beta - 2(\alpha' + \beta') + 10\alpha\beta \right\} \\
- 5\alpha(\alpha' + 3\beta') + 7\beta(5\alpha' + \beta') \\
- 14\alpha'(\alpha' + 6\beta') + 42\alpha\beta(3\alpha' + 2\beta') \\
- 6(7\alpha' + 70\alpha\beta' + 5\beta') + \cdots \right\}, \\
\] (8)

with \( \alpha = b / a' \) and \( \beta = c / a' \). In our experience, for mild orthorhombic anisotropy, one will usually have \( \beta \ll \alpha \), so one can set \( \beta = 0 \) and drop many terms without much loss of accuracy.

The expansion in equation (8) is still not fully separable because of the powers of \( \alpha \) in the denominators of each higher term. We can choose a suitable reference velocity \( v \) and expand

\[ a' = -v^{-2}k_p^{-2} \left\{ 1 - 2E \right\}^{-1} \\
= -v^{-2}k_p^{-2} \left\{ 1 + 2E + 4E^2 + 8E^3 + 16E^4 + \cdots \right\}, \]

where \( k_p^2 = k_1^2 + k_2^2 + k_3^2 \) and \( E = \frac{1}{2} \left( 1 + \frac{a}{v k_p^2} \right) \).

Substituting equation (9) into equation (8) then gives a fully separable series expansion.

The convergence of these series can usually be greatly improved by using pseudo-acoustic approximations. We can change variables to:

\[ \begin{align*}
v_{p1}^2 &= a_{11} \\
v_{p2}^2 &= a_{22} \\
v_{p3}^2 &= a_{33} \\
v_{s1}^2 &= a_{44} \\
v_{s2}^2 &= a_{55} \\
v_{s3}^2 &= a_{66} \\
v_{p1}^2 &= \left[ a_{25}(a_{25} + a_{44}) + a_{44}(a_{25} + a_{33}) \right] / (a_{33} - a_{44}) \\
v_{p2}^2 &= \left[ a_{13}(a_{13} + a_{44}) + a_{44}(a_{13} + a_{33}) \right] / (a_{33} - a_{44}) \\
v_{p3}^2 &= \left[ a_{12}(a_{12} + a_{44}) + a_{44}(a_{12} + a_{33}) \right] / (a_{33} - a_{44}) \\
\end{align*} \]

and then let \( v_{s1} \rightarrow 0 \), \( v_{s2} \rightarrow 0 \), and \( v_{s3} \rightarrow 0 \), giving the pseudo-acoustic orthorhombic Christoffel elements

\[ H_{11} = v_{p1}^2 k_x^2 \\
H_{22} = v_{p2}^2 k_y^2 \\
H_{33} = v_{p3}^2 k_z^2 \\
H_{12} = v_{p1} v_{p3} k_x k_y \\
H_{13} = v_{p1} v_{p2} k_x k_z \\
H_{23} = v_{p2} v_{p3} k_y k_z. \]

This then gives coefficients with the simpler form

\[ \begin{align*}
a &= -\left( v_{p1}^2 \frac{k_y^2}{k_p^2} + v_{p2}^2 \frac{k_z^2}{k_p^2} + v_{p3}^2 \frac{k_x^2}{k_p^2} \right) \\
b &= v_{p1}^2 \left( v_{p2}^2 - v_{p3}^2 \right) k_y^2 k_z^2 + v_{p2}^2 \left( v_{p3}^2 - v_{p1}^2 \right) k_z^2 k_x^2 \\\n&\quad + v_{p3}^2 \left( v_{p1}^2 - v_{p2}^2 \right) k_x^2 k_y^2 \\
c &= v_{p1}^2 \left( v_{p2} v_{p3} \right) + v_{p2}^2 \left( v_{p3} v_{p1} \right) + v_{p3}^2 \left( v_{p1} v_{p2} \right) - v_{p1}^2 v_{p2}^2 - 2v_{p2} v_{p3} v_{p1} \right) k_x^2 k_y^2 k_z^2. \]

The first terms of the separable series expansion then yield the simple P-wave dispersion relation approximation:

\[ \omega' = v_{p1}^2 \frac{k_x^2}{k_p^2} + v_{p2}^2 \frac{k_z^2}{k_p^2} + v_{p3}^2 \frac{k_x^2}{k_p^2} - \frac{v_{p1}^2 \left( v_{p2}^2 - v_{p3}^2 \right) k_y^2 k_z^2}{v_{p1}^2} - \frac{v_{p2}^2 \left( v_{p3}^2 - v_{p1}^2 \right) k_z^2 k_x^2}{v_{p2}^2} - \frac{v_{p3}^2 \left( v_{p1}^2 - v_{p2}^2 \right) k_x^2 k_y^2}{v_{p3}^2}. \]

(13)

Higher order terms become substantially more complicated.

Note that the first series in equation (8) expands around the ellipsoidal term \( a \). The second series in equation (9) then expands the ellipsoid \( a \) as a perturbation \( E \) around a reference sphere with radius \( v_{p1}^2 k_x^2 \), with \( v \) chosen as some average P-wave velocity to get best convergence. We have found that \( v = \sqrt{v_{p1}^2 v_{p2}^2 v_{p3}^2} \) usually works well in practice.

For mathematical convenience so far we have assumed that the vectors normal to the orthorhombic symmetry planes were aligned with the coordinate axes. For general use one needs to handle rotations with arbitrary orientations. Given a coordinate rotation \( \mathbf{\hat{k}} = \mathbf{b} \mathbf{k} \) defined by a 3x3 orthogonal matrix \( \mathbf{b} \), one can rotate the corresponding spatial wavenumbers using the adjoint matrix \( \mathbf{b}^T \mathbf{k} \) and then use the separable approximations above, now in terms of the rotated wavenumbers.
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Example: orthorhombic modeling

Figure 1 shows a time snapshot of point-source modeling in a homogeneous axis-aligned orthorhombic medium using the generalized pseudospectral method of equation (2) based on a separable pseudo-acoustic dispersion approximation. The medium is homogeneous, with parameters (in km/s) $v_{px} = 3.674$, $v_{py} = 4.135$, $v_{pz} = 3.0$, $v_{px1} = 3.550$, $v_{px2} = 2.683$, $v_{px3} = 3.674$, $v_{py1} = 1.368$, $v_{py2} = 1.499$, and $v_{py3} = 1.2$. One can see an accurate P-wave arrival, and complete elimination of shear wave energy. Figure 2 shows the same modeling result, but now in a rotated and tilted orthorhombic medium. Figure 3 shows similar generalized pseudospectral modeling in a complicated model with a complex salt body embedded in tilted and rotated orthorhombic sediments.

Discussion and conclusions

The generalized pseudospectral method described here provides a simple, stable method for modeling and reverse-time migration of P-waves in mild orthorhombic anisotropy. Retaining good accuracy for stronger anisotropy requires additional, more complicated terms that increase the computational expense, and the approximations used here can fail when the anisotropy becomes sufficiently large. Implementation of these mixed-domain algorithms is most practical on computer architectures that efficiently implement large 3D FFTs.

Note that VTI is a special case of orthorhombic, so the pseudo-acoustic approximations here can be reduced to corresponding ones for TI by setting $v_{px3} = v_{py} = v_{pz}$ and $v_{px2} = v_{px1}$. If one additionally chooses $v_{py} = v_{pz}$, equation (13) reduces to Harlan’s (1995) approximation as used by Etgen and Brandsberg-Dahl (2009). However, this will usually be less accurate than choosing an appropriate average reference velocity value.

Note also that equation (2) really implements the leading term in the more accurate operator expansion (Etgen and Brandsberg-Dahl, 2009)

$$u(x, t + \Delta t) = 2 \cos(\sqrt{A} \Delta t) u(x, t) - u(x, t - \Delta t)$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} A^{j/2} \Delta t^j u(x, t) - u(x, t - \Delta t).$$

This operator series can be efficiently approximated by using Chebyshev polynomials (Pestana and Stoffa, 2010), or by using numerically fitted coefficients (Soubaras and Zhang, 2008; Fowler et al., 2010). These higher-order series methods allow using larger time steps, but still depend on separable approximations such as those introduced here to evaluate the operator $\omega^j = A(k, \nu)$.

Finally, note that general triclinic anisotropy also gives a cubic polynomial dispersion relation of the form in equation (3), and that nothing in the derivation of the series expansion in equation (8) depends on the specific form of the Christoffel elements in equation (5). By substituting the general triclinic Christoffel elements instead, one can extend the separable series expansions derived here for use in not just orthorhombic media, but fully general anisotropic media.
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Figure 1: Time snapshot of a wavefield modeled with a generalized pseudospectral method in an axis-aligned orthorhombic medium. The pressure source is in the middle of the grid. We show a horizontal x-y slice through the source location (left), a vertical x-z slice (middle) and a vertical y-z slice (right).

Figure 2: Time snapshot of a wavefield modeled with a generalized pseudospectral method in a rotated and tilted orthorhombic medium. The pressure source is in the middle of the grid. We show a horizontal x-y slice through the source location (left), a vertical x-z slice (middle) and a vertical y-z slice (right).

Figure 3: Time snapshot of a wavefield modeled with a generalized pseudospectral method in a 3D model with a complex salt body embedded in highly heterogeneous tilted and rotated orthorhombic sediments.
EDITED REFERENCES
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REFERENCES


