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Tomographic Velocity Model Building Using Iterative Eigendecomposition

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SUMMARY

Tomographic velocity model building has become an industry standard for depth migration. However, regularizing tomography still remains a subjective virtue, if not black magic. Singular value decomposition (SVD) of a tomographic operator or, similarly, eigendecomposition of corresponding normal equations, are well known as a useful framework for analysis of most significant dependencies between model and data. However, application of this approach in velocity model building has been limited, primarily because of the perception that it is computationally prohibitively expensive for the contemporary actual tomographic problems. In this paper, we demonstrate that iterative decomposition for such problems is practical with the computer clusters available today, and as a result, this allows us to efficiently optimize regularization and conduct uncertainty and resolution analysis.
Introduction

Velocity model building is one of the most challenging problems in modern depth imaging. The necessity to account for anisotropy in seismic velocities complicates this issue even further. 3D tomographic analysis became the key technology to tackle this problem (Woodward et al., 1998). However, the tomographic inverse problem is ill-posed, which leads to big uncertainties and ambiguities in the reconstruction. To circumvent this, various forms of regularization are used. In particular, truncated singular value decomposition (SVD) could be used as a form of regularization (Lines and Treitel, 1984; Scales et al., 1990). In general, SVD provides a benign framework for analysis of most significant dependencies between model and data and of resolution and uncertainty for the estimates. For large datasets when direct SVD solution is not computationally feasible, various iterative schemes for partial SVD have been proposed, pioneered by Lanczos. Because the properties of iterative decompositions for symmetric positive-definite matrices are better known than for non-square matrices, instead of computing the SVD of the tomographic operator, one can do eigendecomposition of corresponding symmetric positive-definite matrix of the normal equations. We refer the reader to Saad (2003) and Meurant (2006) for an extensive review on modern iterative methods for sparse matrices and the Lanczos method.

LSQR solvers based on Lanczos iterations became a popular tool for solving the system of linear tomographic equations. Various extensions of LSQR solvers for eigendecomposition in application to covariance and resolution analysis in seismic tomography (primarily for global seismology) have been proposed (Zhang and McMechan, 1995; Berryman, 2000; Minkoff, 1996; Yao et al., 1999; Vasco et al., 2003). While it is argued in Nolet et al. (1999) that such approaches do not span the full eigenspace with the limited number of LSQR iterations, and therefore, the corresponding covariance and resolution analysis is inadequate, Zhang and Thurber (2007) showed that Lanczos iterations with random starting vector can overcome this limitation.

In this paper, we took an approach similar to Zhang and Thurber (2007) and applied it to tomographic velocity model building. Our method is a modification of preconditioned regularized least squares and based on the eigendecomposition of Fisher information matrix. We investigate the process of tomographic regularization and resolution quantification using the apparatus of eigendecomposition.

Preconditioned regularized least squares

We consider the following linearized forward problem of seismic tomography

\[ \Delta y = A \Delta x + n, \] (1)

where \( \Delta y \) and \( \Delta x \) are data and model perturbations, respectively, and \( n \) is the additive noise. \( A \) is the linear operator obtained from the Frechet derivatives of the non-linear tomography operator that models the sensitivity of the picks to the velocity (and optionally to anisotropic parameters). We refer the reader to Woodward et al. (1998) and references within for the details and linearization of the forward problem of common image point (CIP) tomography.

A regularized least squares solution is one of the most common ways to solve equation 1. Its preconditioned version, according to Woodward et al. (1998), can be formulated by first preconditioning the model by \( \Delta x = P \Delta x_P \), and then minimizing the cost functional

\[ \Phi(\Delta x_P) = (\Delta y - AP \Delta x_P)^T D^{-1} (\Delta y - AP \Delta x_P) + \lambda^2 \Delta x_P^T \Delta x_P. \] (2)

Here, \( D \) is the noise covariance matrix; and \( \lambda \) is the damping parameter that controls the trade-off between minimizing the data misfit and minimizing the preconditioned model norm.

Taking the variational derivative of \( \Phi(\Delta x_P) \) with respect to \( \Delta x_P \) and equating the result to zero, the preconditioned regularized solution \( \Delta x_P \) can be written as \( \Delta x_P = \frac{1}{\lambda \Delta x_P^T P D^{-1} P^T A^T} P^T A^T D^{-1} \Delta y \), with \( \Delta x_P := P \Delta x_P + \lambda^2 I \), \( P := P^T A^T D^{-1} A P \) being the Fisher information matrix corresponding to the preconditioned model parameterization \( \Delta x_P \), and \( I \) denotes the identity matrix.
We recognize $\Delta x_p$ as the solution of the following augmented system of linear equations:

$$
\begin{pmatrix}
D^{-1/2} A P & P \nabla
\end{pmatrix} \Delta x_p = \begin{pmatrix}
D^{-1/2} \Delta y
\end{pmatrix}.
$$

(3)

In practice, LSQR methods are used to solve equation 3. The choice of $\lambda$ and $P$ is subjective. It is either an educated guess or determined by a user after testing a range of possibilities. Moreover, as we mentioned in the introduction, the resolution and uncertainty analysis for LSQR is questionable.

**Truncated iterative eigendecomposition**

Now let us consider the objective function in equation 2 for $\lambda = 0$, i.e. preconditioned likelihood function. Then $\Phi(\Delta x_P)$ obtains its minimum at $\Delta x_P = P^{-1} P^T A^T D^{-1} \Delta y$.

As an alternative to the preconditioned regularized least-squares method described in the previous section, one can perform a truncated eigenvalue decomposition of $P^{-1} P^T A^T D^{-1}$ by utilizing finite number of eigenvalues and eigenvectors of $P^{-1} P^T A^T D^{-1}$, i.e.

$$
\tilde{P}_{n} = P_{n} \Lambda_{P_{n}}^{-1} U_{P_{n}}^T,
$$

where $\Lambda_{P_{n}}^{-1}$ is the diagonal matrix formed by the inverses of the $n$ largest eigenvalues of $P^{-1} P^T A^T D^{-1}$, and $U_{P_{n}}$ is the matrix formed by the corresponding $n$ eigenvectors of $P^{-1} P^T A^T D^{-1}$.

The truncation ($n$) and damping ($\lambda$) may be chosen to give approximately the same results (Scales et al., 1990). Damping optimization requires computing $\tilde{P}_{n+1}$, or equivalently solving (3), for a set of $\lambda$. On the other hand, iterative eigendecomposition makes it possible to compute a solution for $\tilde{P}_{n+1}$ by incorporating a new eigenvalue and eigenvector to the eigendecomposition of $\tilde{P}_{n}$, without requiring computation of the eigendecomposition from scratch. This property makes truncated eigenvalue decomposition appealing for the process of regularization optimization, even though eigendecomposition normally requires more iterations and some extra computations than one run of LSQR iterations. Furthermore, automated cutoff criteria for truncation may be developed based on the statistics of the residual updates as we compute more eigenvalues and eigenvectors.

The resolution matrix for $\Delta x_p$ is defined by $R_P := \tilde{P}_{n+1}^T \tilde{P}_n$. Eigendecomposition $P^{-1} P^T A^T = U_{P_{n}} \Lambda_{P} U_{P_{n}}^T$ provides a straightforward way to calculate the resolution matrix:

$$
R_P = \left[U_{P_{n}} \Lambda_{P_{n}}^{-1} U_{P_{n}}^T\right] U_{P_{n}} \Lambda_{P} U_{P_{n}}^T = U_{P_{n}} U_{P_{n}}^T = \tilde{P}_{n+1}^T \tilde{P}_n.
$$

(4)

**Case Study**

The following example illustrates application of truncated iterative eigendecomposition for regularization and resolution analysis using a sample field dataset. Figure 1 shows slices of 3D tomographic velocity update, a diagonal and a row of the resolution matrix for 100 eigenvectors (Figure 1, left column), and 180 eigenvectors (Figure 1, right column). The row of resolution matrix is taken for the node at the cross section of the slices. Figures 1(g) and (h) show the 100th and 180th eigenvector, respectively. The discretization of the volume of interest is 98x170x88 nodes. The free surface corresponds to $z = 0$. A preconditioner with smoothing lengths 10x10x4 nodes was applied. Lanczos iterations with full Gram-Schmidt orthogonalization were used. Iterations were automatically stopped at the 1288th iteration after the loss of orthogonality in single precision as an indication of a good approximation for spanning the whole eigenspace.

The solution for 100 and 180 eigenvectors are presented in Figures 1 (a) and (b), respectively. Since the resolution is a function of illumination, data and modeling errors, preconditioning and truncation, the behavior of the resolution matrix as a function of number of iterations could be used as a truncation criterion. In our case, the results presented in Figures 1 (c), (d), (e) and (f) allow us to judge the quality of tomographic reconstruction between the solutions for 100 and 180 eigenvectors. Although, the solution for 180 eigenvectors (Figure 1 (b)) has more visual
details and higher amplitudes than the solution for 100 eigenvectors (Figure 1 (a)), the solution for 100 eigenvectors is more reliable than that of for 180 eigenvectors because of the smaller spread of the resolution. Furthermore, the eigenfunction in Figure 1(h) has some unphysical edge anomalies.

**Conclusions**

Tomography is an important tool in the modern industry practice for velocity model building. However, a great care should be taken in regularizing the solution. The proper choice of regularization parameters is a big challenge in general. Furthermore, control over the solution uncertainty is of great importance in the inverse problem. In this paper we revisited a long-known eigendecomposition approach, and demonstrated that its iterative implementation is feasible on modern parallel computer architectures for the actual exploration seismic problems. This approach provides extra capabilities for regularization optimization and resolution analysis.

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**References**


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Figure 1: Tomographic velocity update using (a) 100 and (b) 180 eigenvectors. Diagonal of the resolution matrix (see equation 4) for (c) 100 and (d) 180 eigenvectors. Row of the resolution matrix for the velocity node with coordinates \(x = 85, y = 151, \) and \(z = 7\) using (e) 100 and (f) 180 eigenvectors. Eigenvectors (g) 100th and (h) 180th.